The Astrophysical Journal, Vol. 152, May 1968

DYNAMICAL MODELS OF PLANETARY NEBULAE

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Received September 13, 1967; revised October 17, 1967

ABSTRACT

Dynamical models of spherically symmetric, optically thin planetary nebulae have been constructed utilizing an accurate numerical scheme and taking account of the thermal behavior of the gas in detail. The evolution of several model nebulae is exhibited and conclusions are drawn regarding some initial conditions that must prevail in the ejected shell.

I. INTRODUCTION

During the three decades following the pioneering research of Menzel and his co-workers, planetary nebulae have been the subject of a great amount of observational and theoretical research. In several respects these objects are ideally suited for theoretical study. The relative intensities of the hydrogen emission lines (Balmer decrements) have been interpreted with outstanding success by adopting the simplest possible nebular model, consisting of a uniform, quiescent, isothermal medium in which the energy gains per cubic centimeter per second, resulting from the absorption of ultraviolet quanta from a high-temperature exciting star, are balanced by the energy loss rate per cubic centimeter by atomic processes. In reality, however, planetary nebulae such as NGC 7293 exhibit density stratifications that can be allowed for only in an average way in the above model.

Sofia (1967) has shown that, if a nebula is taken to be a quiescent dilute gas of constant density, in which the divergence of the energy flux is zero, then a thermally stable, non-uniform temperature distribution will be established, owing to the variation of the dilution factor with radial distance from the central star. However, this model is unrealistic for planetary nebulae, since planetaries expand into the interstellar gas at supersonic speeds. Quite aside from the existence of a shock front and density gradients, which are not considered in Sofia's models, the divergence terms, resulting from the expansion of the shell, would lead to an adiabatic cooling of the gas which cannot be allowed for properly in any static model. Also, for any reasonable set of assumptions about the central star, Sofia's models (1966) predict electron temperatures in the observable regions of the model nebulae which are about a factor of 2 higher than the observed electron temperatures in planetary nebulae.

Since the physical processes occurring in planetary nebulae have been studied in detail, it would seem justifiable at the present time to attempt to construct dynamical models of these objects which include these processes in the thermal behavior of the gas. Further motivation for the construction of detailed nebular models stems from the fact that high-quality observations of line profiles and Hβ emission profiles are now available for some nebulae.

Mathews (1966) built the first dynamical models of planetary nebulae. However, his models can be objected to on the grounds that (*a*) the thermal behavior of the gas was treated in a very approximate way and (*b*) the resulting comparison of his models with observation was not particularly good. In this paper we will describe new models of planetary nebulae, constructed using an accurate numerical code and including a detailed treatment of the thermodynamics of the nebular shell. These models are in good agreement with the available observational evidence.

II. NUMERICAL PROCEDURES

Rather than adopting the usual Lagrangian approach the problem was formulated in Eulerian variables. This was done in order to allow for future generalization of the numerical code to multidimensional problems. An additional advantage of the Eulerian formulation is that, owing to the constant spacing of the space mesh points, higher-order differences can be utilized readily in calculating space gradients. However, Eulerian variables suffer one great disadvantage over Lagrangian variables in that integral checks, such as energy conservation, are cumbersome to handle numerically and require a great deal of computer time. Accordingly, in testing our code we used the alternative approach of checking its solutions in limiting cases where analytical solutions exist (i.e., free-fall collapse, linearized sound waves, etc.). In all these cases at least six-figure agreement was obtained between the numerical results and the known analytical solutions. Whenever numerical instabilities (due to improper choice of integration steps or initial conditions) arose, they were easy to detect, owing to the fact that the numerical solutions would blow up within a few time steps following the onset of such an instability.

Assuming radial symmetry and including artificial viscosity (Richtmyer 1957), the dynamical equations in Eulerian variables governing the evolution of a model planetary nebula are

$$\frac{\partial \ln y}{\partial \tau} = -u\frac{\partial \ln y}{\partial \xi} - \frac{\partial u}{\partial \xi} - \frac{2u}{\xi}, \tag{1}$$

$$\frac{\partial u}{\partial \tau} = -u\frac{\partial u}{\partial \xi} - \frac{3z}{y}\frac{\partial \ln z}{\partial \xi} - l^2\frac{\partial u}{\partial \xi}\left(2\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial u}{\partial \xi}\frac{\partial \ln y}{\partial \xi}\right) - \frac{K_1}{\xi^2}, \tag{2}$$

and

$$\frac{\partial \ln z}{\partial \tau} = -u\frac{\partial \ln z}{\partial \xi} - \frac{5}{3}\left(\frac{\partial u}{\partial \xi} + \frac{2u}{\xi}\right) + \frac{K_2}{z}(H - C) + \frac{10 l^2}{9}\frac{y}{z}\left(\frac{\partial u}{\partial \xi}\right)^2\left(\frac{\partial \ln y}{\partial \tau} + u\frac{\partial \ln y}{\partial \xi}\right), \tag{3}$$

where the density $\rho = y\rho_0$, pressure $p = zp_0$, velocity $V = (3\rho_0/5p_0)^{1/2}u$, time $t = R(3\rho_0/5p_0)^{1/2}\tau$, space coordinate $r = \xi R$, where R is a characteristic length, $K_1 = 3GM_0\rho_0/5R$, where M_0 is the mass of the central star, $K_2 = 2R(\rho_0/15)^{1/2}p_0^{3/2}$, H is the rate of heat input per cubic centimeter in the gas, C is the cooling rate of the gas per cubic centimeter, and l is an artificial viscosity parameter measuring, roughly, the number of space mesh points over which a shock front is to be artificially smoothed. If $\partial u/\partial \xi < 0$, $l = 3\Delta\xi$, where $\Delta\xi$ is the space interval; if $\partial u/\partial \xi \geq 0$, $l = 0$. In the above definitions, argument 0 refers to the values of the pressure and density in an undisturbed, uniform, interstellar medium surrounding the star prior to the ejection of the planetary nebula. In equation (2), the self-gravitation of the nebular shell has been ignored in comparison with the gravitational influence of the central star. A discussion of the thermodynamic functions H and C will be presented later in this section.

The above system of equations was solved numerically with high precision by a predictor-corrector method utilizing fifth-order backward differences in the time derivatives of the variables. For example, the finite difference equations of continuity are

$$\frac{d \ln y_j^n}{d\tau} = -u_j^n\frac{d \ln y_j^n}{d\xi} - \frac{du_j^n}{d\xi} - \frac{2u_j^n}{\xi}, \tag{4}$$

where superscript n denotes the time step and subscript j indexes the space mesh points. The space gradients were computed accurately using up to fifth-order central differences in the variables.

No attempt was made to treat the dynamics of the gas in the region interior to 3×10^{16} cm ($\xi < 0.005$). Instead, the inner boundary condition was adopted that no matter be allowed to flow across a boundary located at $r = 3 \times 10^{16}$ cm unless the density at the boundary fell below 10 electrons cm^{-3}. When this occurred, enough matter was allowed to flow across the boundary to raise the boundary electron density to 10 cm^{-3}. Interior to the boundary it was assumed that the density had a constant value equal to the density at the boundary and that the spatial dependence of the velocity was linear between the origin and the boundary. In some respects this boundary condition is similar to that at the surface of the central star after mass ejection has ceased. The flow of matter, introduced to avoid numerical difficulties, corresponds to an intermittent solar wind. However, the amount of material that is introduced by this procedure is small enough so that it does not affect the solutions. The "boundary" was placed at $\xi = 0.005$ to minimize numerical difficulties associated with the convergence of spherical waves toward the origin. Experiments were conducted, placing the "boundary" at different (small) radii and it was concluded that this boundary condition has little effect on the dynamics of the nebula as a whole.

Initially, fifty space zones were employed. However, points were added at the outer boundary as the shell expanded with the result that the variables were being advanced at a few hundred space points by the end of long machine runs.

The H function, representing the heating rate per cubic centimeter by ultraviolet quanta from the central star, and the C function, representing the cooling rate per cubic centimeter by atomic processes (forbidden lines and free-free emission), were calculated following the procedures described by Sofia (1966, 1967). In order to calculate these functions, information must be available about the effective temperature, radius, energy distribution, and evolution of the central star. The available models for the flux distributions in the central stars in planetary nebulae (Gebbie and Seaton 1963; Böhm and Deinzer 1965, 1966) do not depart markedly from black bodies. Moreover, our studies show that dynamical models of planetary nebulae are insensitive to the departures from black-body emission found by the above authors. Therefore, black-body flux distributions were assumed for the central stars in the models described in this paper. When evolutionary effects were allowed for, O'Dell's (1962, 1963) models were used. The model nebulae were assumed to consist of fifty-nine ions of the elements H, He, O, N, C, Ne, Mg, Si, Fe, and S in the abundances quoted by Aller (1963), having ionization potentials $\chi < 200$ eV. Forty forbidden transitions were considered, including fine-structure transitions and allowance was made for collisional de-excitation. (For details see Sofia 1966, 1967.) It was assumed throughout that the model nebulae are optically thin. In computing the cooling function, the steady-state ionic balance was calculated at every mesh point using the hydrogenic approximation for the absorption coefficients.

In advancing the variables in time two corrector operations were used, and the τ step was varied according to an empirical Courant-type condition consistent with both numerical accuracy and numerical stability. In practice, the criterion adopted was that, if the current τ step exceeded $0.217\Delta\xi/(|u| + 1)$ at any space mesh point, the τ step was halved and conversely. Since the predictor-corrector algorithm is not self-starting, initializing values were developed for the first five τ steps in the variables by specifying the run of pressure, density, and velocity at $\tau = 0$ and utilizing a simplified algorithm similar to that described by Richtmyer (1957). For details of this procedure see Hunter (1967). As an outer boundary condition it was assumed that the variables remained undisturbed until the first precursors of the shock front were detected, at which time more space mesh points were added. Thus the outer boundary always corresponded to the unperturbed state of the gas.

III. DESCRIPTION OF THE MODELS

No details are known about the mass ejection mechanisms responsible for the formation of the initial nebular shell. Thus the starting conditions for the numerical code cannot be supplied either from theoretical considerations or from observation. However, even if realistic models of the mass ejection were available, a great amount of effort would have to be expended in order to follow the early evolution of the shell, owing to its proximity to the center of the space mesh. Accordingly, we have adopted a procedure similar to that followed by Mathews (1966). All cases were initialized when the back of the shell was 3×10^{16} cm from the central star. Assuming an expansion speed of 20 km sec^{-1}, this represents the location of the shell about 500 years after the mass ejection ceased. The initial density profile of the shell was assumed to conform to a smooth, peaked function of the form

$$y = z = 1 + 10^5 a^{-(\xi - 0.1/\xi - 1.1)}\,, \qquad a = 0.00043945\,. \tag{5}$$

The resulting nebular mass is $\sim 0.5\ M_\odot$, about twice the mean nebular mass suggested by O'Dell (1963). While the above function was assumed partially for numerical convenience, the fact that it is continuous renders it a more plausible initial distribution than the discontinuity assumed by Mathews (1966). An initial temperature of 10^4 ° K was assumed to prevail throughout the shell. On observational grounds, it was necessary to constrain the average expansion speed of the shell to be around 20 km sec^{-1}. However, the distribution of velocity within the shell was left arbitrary. Although a large number of model nebulae were constructed (not necessarily conforming to the initial conditions quoted above), the essential features of all our models are contained in three examples which we now consider in detail.

Following O'Dell (1962, 1963), in Model I it was assumed that a central star of 1 solar radius had a temperature of 40000° K at the time of mass ejection. Thereafter, in a period of 25000 yr, the temperature of the star was assumed to increase linearly to 150000° K, and the radius of the star was assumed to decrease linearly to $10^{-2}\ R_\odot$. The initial shell was assumed to have a constant velocity of expansion of 20 km sec^{-1}. The results of this case are summarized in Figures 1–5.

Comparison with other cases indicated that the results were so insensitive to the evolutionary effects of the central star that it was decided to use non-evolving stars for the remaining models rather than other evolutionary sequences (Harman and Seaton 1964, etc.).

In Model II the exciting star was assumed to be a superluminous giant of 1 solar mass and 10 solar radii, having an effective temperature of 150000° K. The initial velocity profile was assumed to be linear, ranging from 10 km sec^{-1} at the inner boundary of the shell to 30 km sec^{-1} at the outer boundary. An inspection of Figure 6, representing the Hβ emission, shows that this model does not resemble a planetary nebula. Figure 7 shows that the main effect of a superluminous central star is to speed up the heating of the shell following its initial expansion cooling.

Finally, in Model III, the exciting star was assumed to have 1 solar mass, 1 solar radius, and a temperature of 75000° K, and the initial velocity profile *decreases* linearly from 25 km sec^{-1} at the inner boundary to 10 km sec^{-1} at the outer boundary. The main results of this case are summarized in Figures 8–10. In all respects this model resembles observed symmetric planetary nebulae.

IV. CONCLUSIONS

On the basis of our dynamical models of planetary nebulae, we suggest that the following general conclusions be drawn.

a) Irrespective of the initial conditions assumed for the ejected shell, a shock wave always develops in the forward direction. For some initial conditions large-scale mass

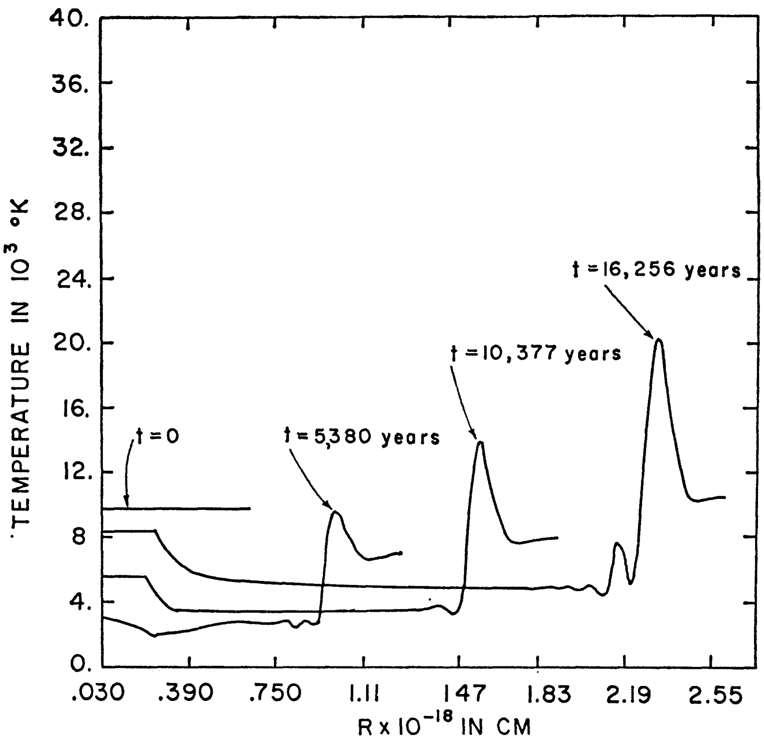

FIG. 1.—Temperature versus space coordinate in Model I

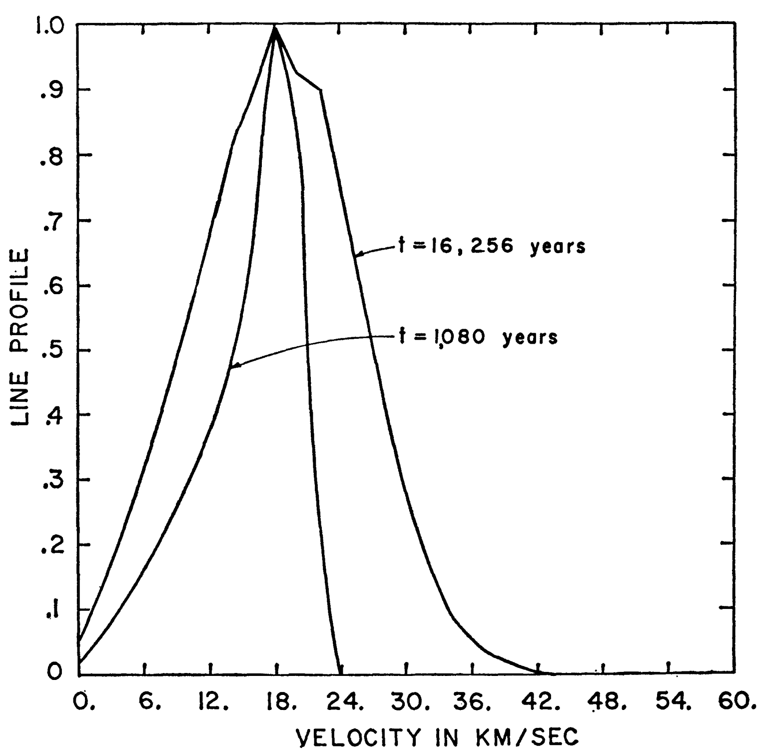

FIG. 2.—Hβ line profile for Model I

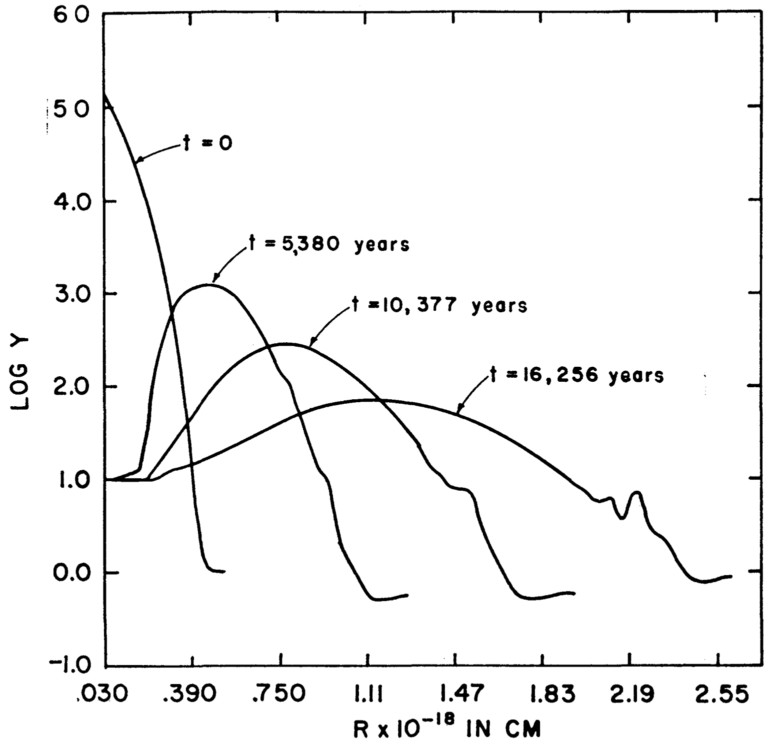

FIG. 3.—Density (electrons/cm³) versus space coordinate in Model I

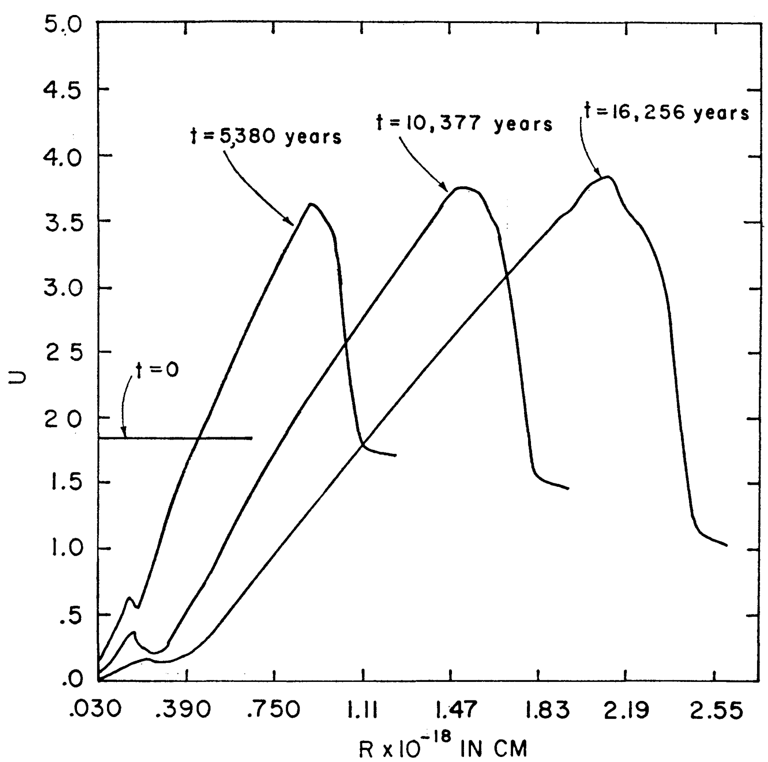

FIG. 4.—u versus space coordinate in Model I

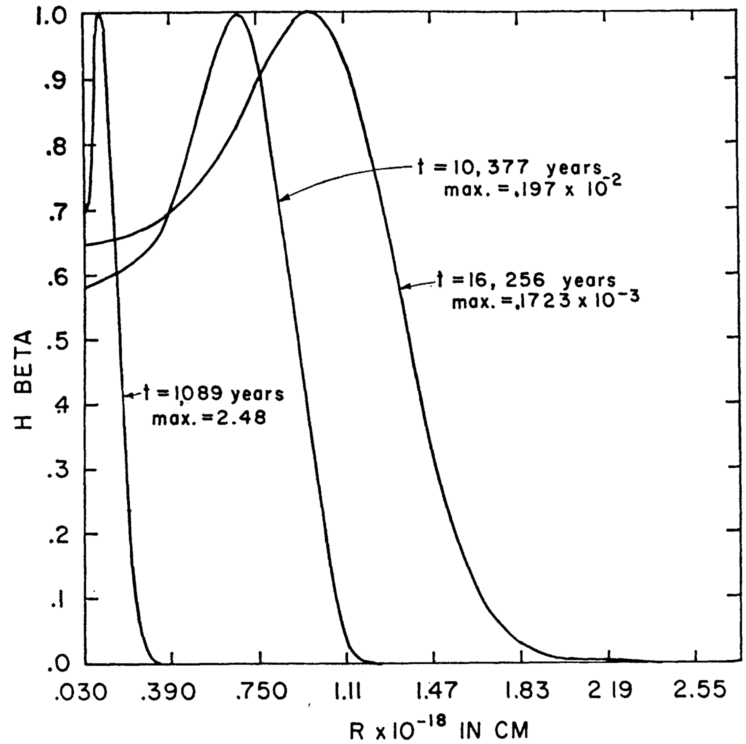

FIG. 5.—Normalized Hβ emission versus space coordinate in Model I. The maximum emission rate (erg/cm³ sec) is indicated.

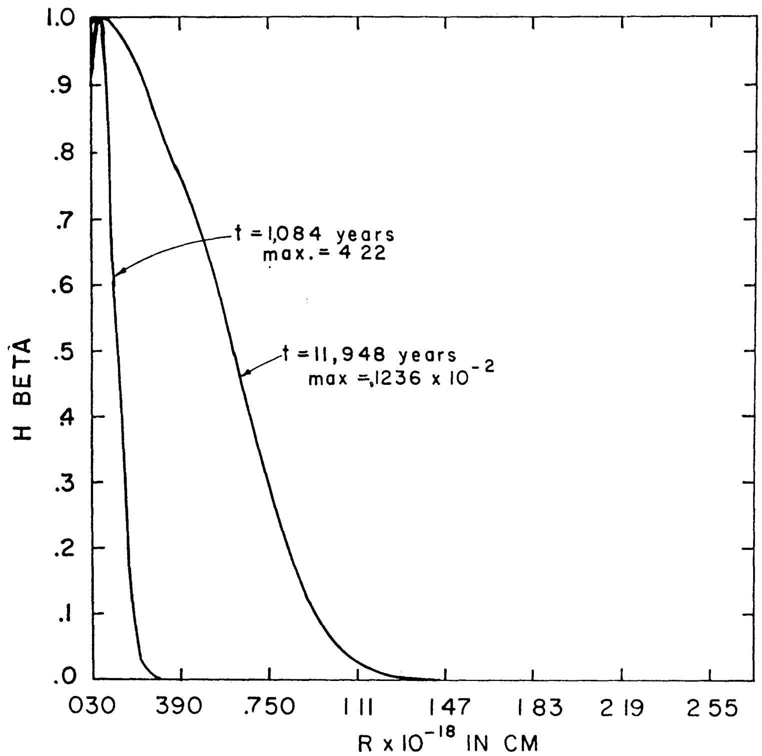

FIG. 6.—Same as Fig. 5, for Model II

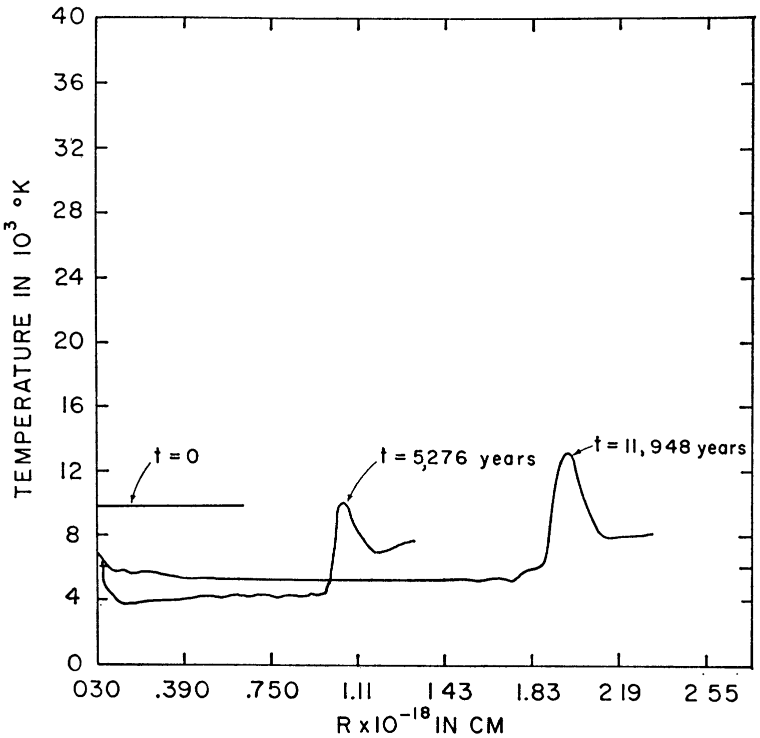

FIG. 7.—Temperature versus space coordinate in Model II

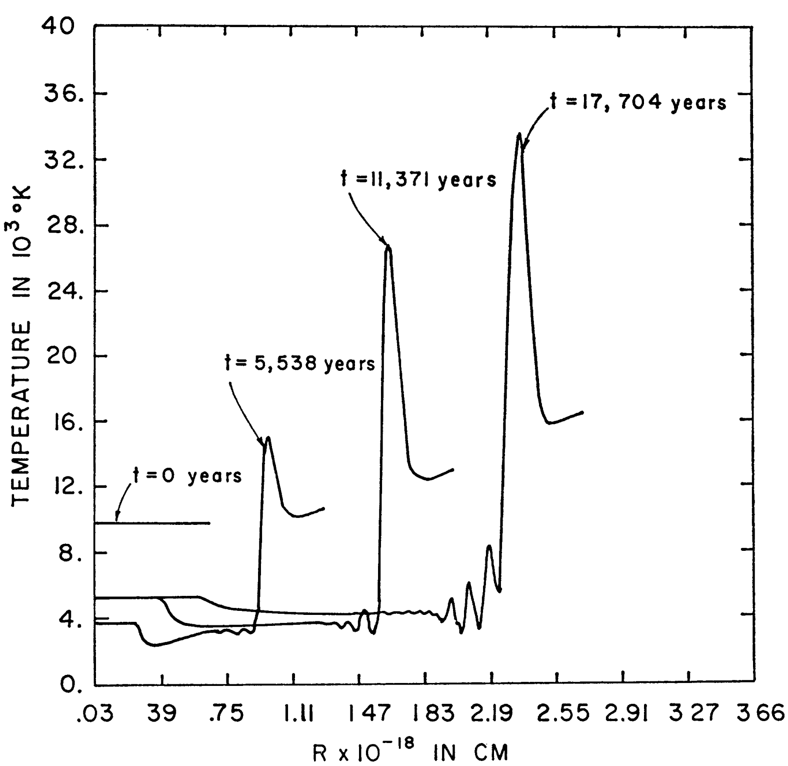

FIG. 8.—Same as Fig. 7, for Model III

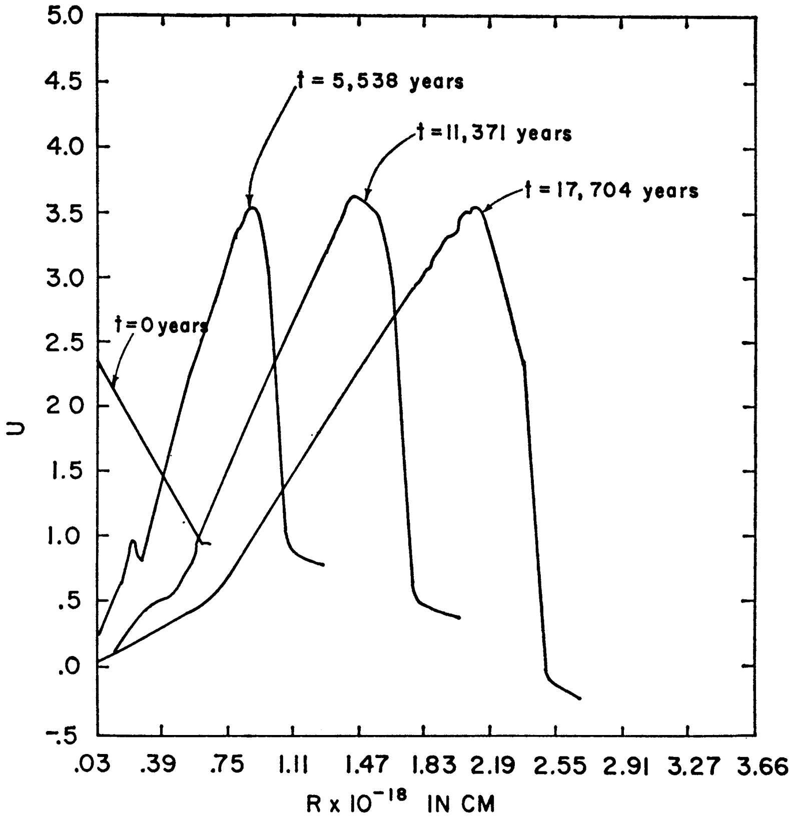

FIG. 9.—u versus space coordinate in Model III

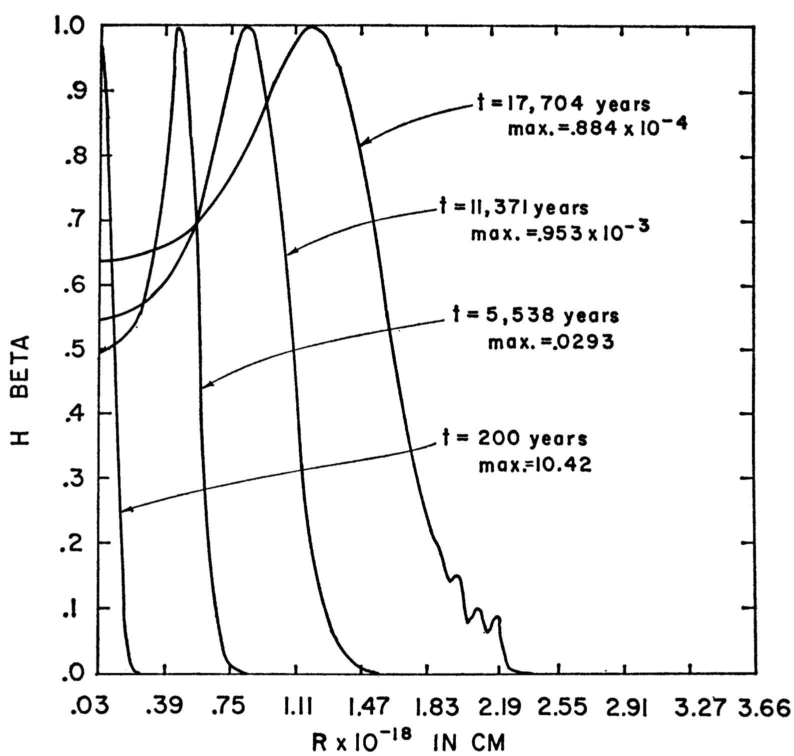

FIG. 10.—Normalized Hβ emission versus space coordinate in Model III

motions develop back toward the central star. Our studies show that the steeper the assumed pressure gradients behind the shell, the larger are the velocities of mass motion toward the center. In this respect our conclusions are consistent with those of Mathews (1966) who showed that, in the absence of pressure support behind an initially discontinuous shell, matter flows back to the central star. However, in all cases studied in this investigation having no pressure discontinuities or large velocity gradients in the initial shell, no substantial amount of matter flowed back onto the central star. Therefore, it would seem that Mathews' conclusion that internal pressure support is necessary in order for the nebulae to maintain a planetary-like appearance is a direct consequence of his adopted initial conditions for the shell. For *numerical reasons* we found it desirable to prohibit the density behind the shell from falling below a prescribed limit (10 electrons/cm^3) in order to prevent a near-perfect vacuum from developing. While physically this corresponds to an outflow of matter from the central star, the role played by the pressure exerted by the matter behind the shell does not appear to be crucial to the problem.

b) Starting with a shell of constant temperature, say, 10^4 ° K, the expansion quickly cools the material to temperatures much below those that would be predicted on the basis of a static equilibrium model. However, once the shell has expanded to a certain degree, the photo-ionization heating dominates over the expansion cooling and the electron temperature rises slowly toward the above-mentioned equilibrium, which is achieved in general when the nebula is too dilute to be seen. This conclusion may explain why the equilibrium temperatures for nebulae derived on the basis of *static* models are in excess of the measured electron temperatures if reasonable parameters are assigned to the exciting star and chemical composition of the nebula. The initially cool and dense shell, which would exist during the early post-ejection phases, would not be observed as a planetary nebula. Essentially the only lines visible in the spectrum would be hydrogen in emission, although a few fine-structure lines should be observable in emission (from above Earth's atmosphere) in the infrared. It may be significant that, while shell stars and emission line stars are observed, no structures resembling planetary nebulae in early stages of expansion have ever been observed.

c) By the time the models had evolved sufficiently to be classified as planetary nebulae, in all cases the velocities of mass motion increased with radial distance from the center. This result is in agreement with Wilson's (1958) studies on the broadening of emission lines having different excitation potentials.

d) No large positive velocity gradients can exist in the initial shell unless pressure support is subsequently supplied from within. Otherwise, the resulting structures do not resemble planetary nebulae (see Fig. 6). If the initial velocity gradient is assumed to be negative in the shell (Model III), as the model evolves it becomes essentially indistinguishable from a case in which the initial velocity gradient is zero. These conclusions are relevant to considerations of mass ejection mechanisms for the initial shell.

e) The computed line profiles (including Doppler broadening) for the model planetary nebulae closely resemble those observed in symmetric planetary nebulae (Osterbrock, Miller, and Weedman 1966). Moreover, the Hβ emission isophotes, predicted on the basis of our model nebulae, are similar to the Hβ emission observed in planetary nebulae (Aller 1956). Thus it is unnecessary to postulate the existence of torus-like structures.

f) The evolution of the central star appears to have little effect on the structure of planetary nebulae.

It is a pleasure to acknowledge helpful comments from Drs. J. S. Mathis, W. G. Mathews, and R. Stothers. A special mention is due to Miss F. Nabi for her very able assistance in the numerical calculations. Most of the research was carried out while one of us (S. S.) held a National Research Council Postdoctoral Research Associateship supported by the National Aeronautics and Space Administration. Finally, we want to thank Dr. R. Jastrow for his hospitality at the Institute for Space Studies and for making available to us the very substantial amount of computer time required to do this problem.

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